Solution 3

1. Normly convergence implies weakly convergence, thus $\overline{E} \subset \overline{E}^w$.

It's easy to check that $\overline{E}^w$, $\overline{E}$ are subspace, in particular, $\overline{E}$ is closed.

Suppose $\overline{E}^w \setminus \overline{E} \neq \phi$.

· By Hahn-Banach separation Thm, there exists $f \in X^*$ st $f(\overline{E}) = 0$ and $f(x) \neq 0$, for $x \in \overline{E}^w \setminus \overline{E}$.

Let $x_0 \in \overline{E}^w \setminus \overline{E}$, then $\exists$ sequence (x_n) in E st $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Notice that $E \subset \overline{E}$, we have $f(x_n) \equiv 0$, $\forall n \in \mathbb{N}$.

Thus $\lim_{n\to\infty} f(x_n) = 0 = f(x_0)$, which is a contradiction.

Therefore, $\overline{E}^w = \overline{E}$.

2. (i) $\operatorname{Ker} T^* = \{ y^* \in Y^* : T^* y^* = 0 \}$

Since $\langle T^* y^*, x \rangle = \langle y^*, Tx \rangle$,

We have $T^* y^* = 0 \iff y^* \circ T = 0$

Therefore, $\operatorname{Ker} T^* = \{ y^* \in Y^* : y^* \circ T = 0 \}$.

(ii) $\Rightarrow$: If $\overline{TX} = Y$, then T^* is injective.

It's sufficient to prove $\operatorname{Ker}(T^*) = \{0\}$.

By (ii), if $y^* \in \operatorname{Ker}(T^*)$ and $y^* \neq 0$,

Then $y^* \circ T(x) = 0$, $\forall x \in X$. (1)

Since $T(X)$ is dense in Y, for any $y \in Y$, $\exists (x_n) \in X$, st $\lim_{n\to\infty} \| T(x_n) - y \| = 0$.

Combining with (1), we have

$$\|y^*(y)\| = \|y^*(y) - y^*(TX_n)\| \quad , \text{ for } n \in \mathbb{N}$$
$$= \lim_{n\to\infty} \|y^*(y - TX_n)\|$$
$$\leq \|y^*\| \cdot \lim_{n\to\infty} \|y - TX_n\|$$
$$= 0$$

Therefore, $y^* = 0$.

$\Leftarrow$ If T^* is injective then $\overline{T(X)} = Y$.

- $T(X)$ is a subspace of Y

  if $y_1, y_2 \in T(X)$, then $\exists x_1, x_2 \in X$ st $T(x_i) = y_i$, $i=1,2$.

  $y_1 + y_2 = T(x_1 + x_2) \in T(X)$

  For $\lambda \in \mathbb{R}$, $\lambda y_1 = \lambda T(x_1) = T(\lambda x_1) \in TX$.

Suppose $Y \setminus \overline{T(X)} \neq \phi$, assume $y_0 \in Y \setminus \overline{T(X)}$.

By Hahn-Banach separation Thm, $\exists y^* \in Y^*$,

$y^*(\overline{T(X)}) = 0$ and $y^*(y_0) \neq 0$. Thus, $\|y^*\| > 0$.

On the other hand, $y^*(TX) = 0 \Rightarrow y^* \circ T = 0$,

which implies $y^* \in \ker T^*$.

This contracts with $\ker T^* = \{0\}$.

Therefore, $Y = \overline{T(X)}$.